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RISK AVERSION AND THE EFFICIENCY OF FIRST AND SECOND PRICE AUCTIONS

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Summary:

I contrast a first price (discriminatory, Dutch) auction to a second price (competitive, English, Vickrey) auction. Two results are obtained. First, when the seller or the potential buyers are risk averse, the first price auction is more efficient *ex post* than the second price auction. Second, if the buyers are not too risk averse, the first price auction Pareto dominates *ex ante* the second price auction. These results are derived in a setting where (1) the bidders have identical utility functions, (2) the bidders' evaluations of the item are independent and identically distributed *ex ante*, and (3), most crucially, the seller announces beforehave a reservation price which every bid must exceed.

# RISK AVERSION AND THE EFFICIENCY OF FIRST AND SECOND PRICE AUCTIONS

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## 1. Introduction

I contrast a first price (discriminatory, Dutch) auction to a second price (competitive, English, Vickrey) auction. Two results are obtained. First, when the seller or the potential buyers are risk averse, the first price auction is more efficient *ex post* than the second price auction. Second, if the buyers are not too risk averse, the first price auction Pareto dominates *ex ante* the second price auction. These results are derived in a setting where (1) the bidders have identical utility functions, (2) the bidders' evaluations of the item are independent and identically distributed *ex ante*, and (3), most crucially, the seller announces beforehand a reservation price which every bid must exceed.

The logic of the first result is simple. A sale does not occur, because no bid is cast, whenever the highest value among the bidders is less than the reservation price. In the second price auction and often the first price auction, the seller sets a reservation price greater than his actual cost. Consequently, a sale may not occur even though some bidder values the item more than its cost. But the item is sold more frequently in the first price auction, since the seller sets a lower reservation price in the first price auction whenever he or the

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bidders are risk averse. Therefore the first price auction is *ex post* more efficient.

The second result is obtained by showing that if both auctions have the same reservation price, then the seller's random profits in the second price auction are dominated, in the sense of second degree stochastic dominance, by his profits in the first price auction. A conjecture of Vickrey [1961] follows, namely, that a risk averse seller prefers the first price auction. Risk neutral buyers also prefer the first price auction, since a lower reservation price in the first price auction causes the expected sale price to also be lower. Consequently, if the seller is risk averse and the buyers are not very risk averse, every agent prefers *ex ante* the first price auction.

Both results seemingly contradict results obtained in the literature since Vickrey [1961]. The difference is due to the inclusion here of risk aversion and seller reservation prices. I comment further on the literature in the concluding section.

## 2. Framework

The seller conducts an auction to sell one item to one of n possible buyers.[1] The buyers will also be referred to as bidders. Each buyer $i = 1,2,\ldots,n$ values the item at $v_i$. The value $v_i$ is known to i but is regarded as a random variable by the seller and the other buyers. Each buyer regards the values of the other buyers as independently and identically distributed. The seller also regards the bidders' values as independently and identically distributed. All agents attribute the same marginal distribution function $F(v)$ to each value. F is assumed to satisfy the regularity assumption:

A1. (i) F has a continuous density function f;

(ii) $f(v) > 0$ if and only if $v \varepsilon [\underline{v},\overline{v}]$; and

(iii) $\frac{f}{1-F}$ is a strictly increasing function on $[\underline{v},\overline{v}]$.

Assumption A1 (iii) will serve two purposes: (1) it is a sufficient condition for there to be a unique, interior solution to the seller's problem in the second price auction (proposition 1), and (2) since it is a global property, it allows global comparative static (proposition 2) and between-auction (theorem 1) comparisons to be made. Essentially A1 (iii) requires that the density function not decrease too quickly. Most standard distributions satisfy it.

The useful derived probability functions are the density and distribution functions of the maximum value,

$$g_n^1(x) = nF(x)^{n-1}f(x) \quad \text{and} \quad G_n^1(x) = F(x)^n,$$

and the density and distribution functions of the second greatest value,

$$g_n^2(x) = n(n-1)F(x)^{n-2}(1-F(x))f(x) \quad \text{and}$$

$$G_n^2(x) = nF(x)^{n-1} - (n-1)F(x)^n.$$

The buyers all have the same utility function $u_B$ of income. The seller has a utility function $u_S$ of income. Both functions satisfy

A2. $u_B$ and $u_S$ are both concave, strictly increasing, and twice continuously differentiable.

The seller incurs a cost c ($\underline{v} \leq c < \overline{v}$) for providing the item. Hence, if w is the sale price, the seller's utility is $u_S(w-c)$ if the item

is sold and $u_S(0)$ if it is not sold. Similarly, a bidder's utility is $u_B(v_i - w)$ if he purchases the item and $u_B(0)$ otherwise.

The mechanics of the two auctions are as follows. In either auction the seller first publicly sets a reservation price r, which is interpreted as his declared cost.[2] Then any bidder who wishes submits a sealed bid that must not be less than r. If any bid is submitted in the second price auction, the item is sold to the high bidder at a price $w_2$ that is equal to the second highest bid or, if only one bid is submitted, equal to r. If any bid is submitted in the first price auction, a sale is made to the highest bidder at a price $w_1$ that is equal to his own bid.

The subsequent models also apply to two variations of these procedures. First, the bids can be oral rather than sealed. As Vickrey [1961] observes, in this framework the first price sealed-bid auction is strategically isomorphic to an oral Dutch (descending) auction, and the second price sealed-bid auction is isomorphic to an oral English (ascending) auction.

The second variation is that the reservation price need not necessarily be announced. If it is kept secret, but the bidders know of its existence,[3] then the bidders must bid according to what they think it is. When the bidders are in a Nash equilibrium, their expectations about the reservation price (and each others' bidding strategies) will be correct. The seller will be assumed to act like a Stackelberg monopolist, taking into account how bidders' strategies depend on the reservation price when he sets it to maximize his expected utility. Hence the equilibrium notion may still be applicable when the reservation price is not announced.[4]

3. Second Price Auction

The dominant strategy of a bidder in a second price auction is to submit a bid equal to his true value (Vickrey [1961]). An additional proviso is that a bidder will not bid if his value is less than the reservation price. Hence the price $w_2$ is equal to the maximum of the reservation price and the second highest value. Therefore the seller's expected utility from setting a reservation price r is

$$\int_r^{\bar{v}} u_S(x-c)\, g_n^2(x)\,dx + u_S(r-c)\,[G_n^2(r) - G_n^1(r)] + u_S(0)G_n^1(r).$$

The first order condition for maximizing this expression with respect to r reduces to

$$(1) \qquad \frac{u_S'(r-c)}{u_S(r-c) - u_S(0)} = \frac{f(r)}{1 - F(r)} .$$

Assumptions A1 and A2 insure that there is a unique solution $r_2$ to (1), and that $r_2$ solves the seller's maximization problem. Furthermore, A2 implies that the left side of (1) is infinite at $r = c$. Therefore we have

Proposition 1: The seller's optimal reservation price $r_2$ in the second price auction is the solution to (1), is independent of the number of bidders, and is strictly greater than the cost c.[5]

Proposition 1 indicates that the second price auction is not (*ex post*) efficient when the seller can set a reservation price. With positive probability a sale that should occur will not occur, i.e., with positive probability the highest value v will satisfy $c < v < r_2$.

However, the auction is more efficient if the seller is more risk averse. A more risk averse seller sets a lower reservation price.

<u>Proposition 2</u>: Let $u_S$ and $\bar{u}_S$ be two utility functions satisfying A2, with the absolute risk aversion of $\bar{u}_S$ uniformly greater than that of $u_S$. Then a seller with utility function $\bar{u}_S$ sets a lower reservation price in the second price auction than does a seller with the utility function $u_S$.

<u>Proof</u>: The function $\bar{u}_S$ satisfies

$$-\frac{\bar{u}_S''(x)}{\bar{u}_S'(x)} > -\frac{u_S''(x)}{u_S'(x)}$$

for all x, which implies that

$$\frac{\bar{u}_S'(r-c)}{\bar{u}_S(r-c)-\bar{u}_S(0)} < \frac{u_S'(r-c)}{u_S(r-c)-u_S(0)} \tag{2}$$

for all $r > c$ (Pratt [1964]). A2 implies that both ratios in (2) are decreasing functions of r. Let $r_2$(resp. $\bar{r}_2$) be the reservation price of the seller with utility function $u_S$(resp. $\bar{u}_S$). If $\bar{r}_2 \geq r_2$, then (1) and (2) imply

$$\frac{f(\bar{r}_2)}{1 - F(\bar{r}_2)} < \frac{f(r_2)}{1 - F(r_2)} .$$

But this inequality and $\bar{r}_2 \geq r_2$ contradict A1. Hence $\bar{r}_2 < r_2$. □

## 4. First Price Auction

Bidders do not have a dominant bidding strategy in a first price auction. However, viewing the auction as a game of incomplete information,

there is a symmetric "Bayesian equilibrium" in the sense of Harsanyi [1967]. That is, there is a bidding function $b(v,r)$ such that each bidder can do no better than bid $b = b(v,r)$ when he has value $v$, is in a first price auction that has a reservation price $r$, and every other bidder bids in accordance with the function $b(\cdot,r)$. This equilibrium concept has been used for auctions since Vickrey [1961].

Proposition 3: The unique differentiable and increasing equilibrium $b(v,r)$, defined for $v \geq r$, is the solution to the differential equation

$$\text{(3)} \qquad \frac{db}{dv} = (n-1)\,\frac{f(v)}{F(v)}\left\{\frac{u_B(v-b)-u_B(0)}{u_B'(v-b)}\right\}$$

that satisfies $b(r) = r$. Equivalently, $b(v,r)$ is implicitly defined by

$$\text{(4)} \qquad u_B[v-b(v,r)] - u_B(0) = \int_r^v u_B'[x-b(x,r)]\left[\frac{F(x)}{F(v)}\right]^{n-1} dx.$$

Proof: It is relatively well-known that (3) and the initial condition $b(r,r) = r$ characterize the equilibrium, e.g., Ortega-Reichart [1968], Wilson [1977], Holt [1977], Harris and Raviv [1978], and Samuelson [1978] derive similar expressions. Assumptions A1 and A2 can be used to show that $b(v,r)$ actually is an equilibrium. I indicate here only how (3) is derived. The function $b(v,r)$ increases in $v$. Hence, if $\sigma$ is a function defined by $b(\sigma(p),r) = p$, then a bidder who bids $p$ when the others use $b$ has a probability of winning equal to $\Pr(p \leq \max_{i \leq n-1} b(v_i,r)) = F(\sigma(p))^{n-1}$. His expected utility is

$$u_B(v-p)F(\sigma(p))^{n-1} + u_B(0)[1-F(\sigma(p))^{n-1}].$$

Because $b(v,r)$ is an equilibrium, the first order condition for maximizing this expression with respect to $p$ must be satisfied by $p = b(v,r)$. Substitution of $p = b(v,r)$ into the first order condition yields (3) as a necessary condition for $b(v,r)$ to be an equilibrium. The fact that $b(r) = r$ follows from the observation that $b(r) < r$ implies that a bidder with value $v = r$ could do better by bidding greater than $b(r)$ and less than $r$. Expression (4) is derived by writing (3) as

$$-F(v)^{n-1}u_B'(v-b(v,r))b_v(v,r) + (n-1)F(v)^{n-2}f(v)[u_B(v-b(v,r))-u_B(0)]$$
$$+ F(v)^{n-1}u_B'(v-b(v,r)) = F(v)^{n-1}u_B'(v-b(v,r)).$$

Integrating both sides from $r$ to $v$ results in

$$[u_B(x-b(x,r))-u_B(0)]F(x)^{n-1}\Big|_r^v = \int_r^v u_B'[x-b(x,r)]F(x)^{n-1}dx.$$

Because $b(r,r) = r$, this expression is (4). □

Expression (4) implies that a bidder bids strictly less than his value in a first price auction, except when his value is equal to the reservation price. If bidders are risk neutral, then (4) can be solved to yield

$$b^*(v,r) = v - \int_r^v [\frac{F(x)}{F(v)}]^{n-1}dx$$

$$(4^*) \qquad = F(v)^{-(n-1)}\left\{\int_r^v x(n-1)F(x)^{n-2}f(x)dx + rF(x)^{n-1}\right\}$$

$$= \int_{\underline{v}}^v \max\ (x,r)\ \frac{g_{n-1}^1(x)}{G_{n-1}^1(v)}\ dx.$$

Hence a buyer's bid is equal to the expectation of his competitors' values, where the seller is treated as a competitor with value r and the buyer's expectation is taken conditional upon his own value being the greatest.

Another useful implication of proposition 1 is that risk averse bidders bid more than if they were risk neutral. If $u_B'' < 0$, then (3) implies that $b_v(v,r) > b_v^*(v,r)$ at any v for which $b(v,r) = b^*(v,r)$. Since $b(r,r) = b^*(r,r)$, it follows that $b(v,r) > b^*(v,r)$ for all $v > r$.

Now we consider the seller's problem. Because the sale price $w_1$ is equal to $b(v,r)$, where v is the maximum value, the seller's expected utility from setting the reservation price at r is

$$\int_r^{\bar{v}} u_S[b(x,r)-c]g_n^1(x)dx + u_S(0)G_n^1(r).$$

The first order condition for maximizing this with respect to r is

$$(6) \qquad \int_r^{\bar{v}} u_S'[b(x,r)-c]b_r(x,r)g_n^1(x)dx = [u_S(r-c)-u_S(0)]g_n^1(r).$$

Because the left side of (6) is positive, so is the right side. Hence any solution r to (6) is greater than c. Therefore we have

<u>Proposition 4</u>: An optimal reservation price $r_1$ for the seller in the first price auction exists and is either equal to c or satisfies (6) and is strictly greater than c.

5. <u>Ex Post Comparison</u>

Given our simple framework, both auctions result in either the item not being sold or in it being sold to the bidder with the maximum value. Furthermore, the value of a winning bidder in either auction is greater

than the seller's cost. Both auctions are consequently efficient in every state of the world in which a sale occurs in both auctions. The two auctions are equally efficient in any state of the world in which a sale does not occur in both auctions. However, one auction can still be *ex post* more efficient if the set of states of the world in which it results in a sale strictly includes the set of states in which the other auction results in a sale. Theorem 1 shows that the first price auction is more efficient, in this sense, than the second price auction if the seller or the buyers are risk averse. Because risk aversion implies $r_1 < r_2$, the set of states in which a sale occurs in the first price auction, $\{(v_1,\ldots,v_n) | \max v_i \geq r_1\}$, strictly includes the set of states in which a sale occurs in the second price auction, $\{(v_1,\ldots,v_n) | \max v_i \geq r_2\}$.

Theorem 1: Given A1 and A2, there exists r* such that

$$\text{(i)} \quad u''_B = u''_S = 0 \implies c < r_1 = r_2 = r^*,$$ [6]

$$\text{(ii)} \quad u''_B < u''_S = 0 \implies c \leq r_1 < r_2 = r^*, \text{ and}$$

$$\text{(iii)} \quad u''_S < 0 \implies c \leq r_1 < r_2 < r^*.$$

The proof requires a lemma.

Lemma 1: Suppose $\underline{v} < r < \overline{v}$. Then $b^*_r(v,r)F(v)^{n-1} = F(r)^{n-1}$. If $u''_B < 0$, then $0 \leq b_r(v,r)F(v)^{n-1} < F(r)^{n-1}$ for any $v > r$.

Proof: Differentiation of (4*) yields the result for b*. So assume $u''_B < 0$. We first show that $b_r(v,r) \geq 0$. Choose $\overline{r}$ such that $\overline{v} > \overline{r} > r$. Then (4) implies $b(\overline{r},r) < \overline{r} = b(\overline{r},\overline{r})$. Assume there exists some b and $v' > \overline{r}$ such that $b = b(v',r) = b(v',\overline{r})$. Then, because the differential equation (3) has a unique solution satisfying the initial condition

$b(v') = b$, $b(v,\bar{r}) = b(v,r)$ for all $v \geq \bar{r}$. This contradiction implies $b(v,\bar{r}) \neq b(v,r)$ for all $v \geq \bar{r}$. Continuity and $b(\bar{r},\bar{r}) > b(\bar{r},r)$ now imply that $b(v,\bar{r}) > b(v,r)$ for all $v \geq \bar{r}$. Hence $b_r(v,r) \geq 0$.

Now define

$$\alpha(v,b) = \frac{u_B(v-b)-u_B(0)}{u_B'(v-b)} \quad .$$

Note that $u_B'' < 0$ implies that $\alpha_b(v,b) < -1$. Differentiating (3) with respect to $r$ yields

$$b_{vr}(v,r) = (n-1)\,\frac{f(v)}{F(v)}\,\alpha_b(v,b)b_r(v,r).$$

Therefore

$$(5) \qquad \frac{\partial\{b_r(v,r)F(v)^{n-1}\}}{\partial v} = b_{vr}(v,r)F(v)^{n-1} + b_r(v,r)(n-1)F(v)^{n-2}f(v)$$

$$= (n-1)b_r(v,r)F(v)^{n-2}f(v)[\alpha_b(v,b) + 1] \leq 0.$$

Now, $b(r,r) = r$ is an identity in $r$, so that $b_v(r,r) + b_r(r,r) = 1$. Also, (3), $b(r,r) = r$, and $F(r) > 0$ imply $b_v(r,r) = 0$. Hence $b_r(r,r) = 1$. Thus continuity, (5), and $\alpha_b < -1$ imply that $b_r(v,r)F(v)^{n-1}$ is a strictly decreasing function of $v$ on some interval containing $r$. Therefore, for any $v > r$,

$$b_r(v,r)F(v)^{n-1} < b_r(r,r)F(r)^{n-1} = F(r)^{n-1}. \quad \square$$

<u>Proof of Theorem 1</u>: Let $r^*$ be defined as the $r_2$ the seller sets when $u_B'' = u_S'' = 0$. Since $r_2$ is independent of $u_B$, $r_2 = r^*$ if $u_B'' < u_S'' = 0$. Also, proposition 2 implies $r_2 < r^*$ if $u_S'' < 0$. It only remains to show $r_1 \leq r_2$, with equality holding if and only if $u_B'' = u_S'' = 0$.

If $r_1 = c$, then proposition 1 implies $r_1 < r_2$. Hence we can assume $c < r_1$, so that $r_1$ satisfies (6). Substitution of $nF(x)^{n-1}f(x)$ for $g_n^1(x)$ into (6) yields, upon rearrangement,

$$\frac{u_S'(r_1-c)}{u_S(r_1-c)-u_S(0)} = \frac{f(r_1)}{\displaystyle\int_{r_1}^{\bar{v}} \frac{u_S'[b(x,r_1)-c]}{u_S'(r_1-c)}\, b_r(x,r_1) \left[\frac{F(x)}{F(r_1)}\right]^{n-1} f(x)dx} .$$

For any $r_1 < x < \bar{v}$, $b(x,r_1) > r_1$ implies that

$$0 < \frac{u_S'[b(x,r_1)-c]}{u_S'(r_1-c)} \begin{cases} < 1 \text{ if } u_S'' < 0 \\ = 1 \text{ if } u_S'' = 0. \end{cases}$$

Furthermore, lemma 1 implies that

$$0 \leq b_r(x,r_1) \quad \frac{F(x)}{F(r_1)}^{n-1} \begin{cases} < 1 \text{ if } u_B'' < 0 \\ = 1 \text{ if } u_B'' = 0. \end{cases}$$

Therefore, if $u_B'' < 0$ or $u_S'' < 0$, then

$$\frac{u_S'(r_1-c)}{u_S(r_1-c)-u_S(0)} > \frac{f(r_1)}{\displaystyle\int_{r_1}^{\bar{v}} f(x)dx} = \frac{f(r_1)}{1-F(r_1)} ,$$

and the inequality is an equality if $u_B'' = u_S'' = 0$. If $u_B'' < 0$ or $u_S'' < 0$, and if $r_2 \leq r_1$, then

$$\frac{f(r_2)}{1-F(r_2)} = \frac{u_S'(r_2-c)}{u_S(r_2-c)-u_S(0)}$$

$$\geq \frac{u_S'(r_1-c)}{u_S(r_1-c)-u_S(0)} > \frac{f(r_1)}{1-F(r_1)} ,$$

a contradiction of $\frac{f}{1-F}$ strictly increasing. Hence $r_1 < r_2$ if $u_B'' < 0$ or $u_S'' < 0$. A similar argument shows that $r_1 = r_2$ if $u_B'' = u_S'' = 0$. □

This section concludes with an example based on the benchmark distribution, the uniform. Let $F(x) = x$ on the interval $[0,1]$. Let $u_B(x) = x^a$ $(0 < a \leq 1)$ and $u_S(x) = x^s$ $(0 < s \leq 1)$. Let $m = 1 + (n-1)a^{-1}$. Then simple calculation yields

$$b(v,r) = v[(\frac{m-1}{m}) + (\frac{1}{m})(\frac{r}{v})^m].$$

The optimal reservation price in the second price auction is

$$r_2 = \frac{c+s}{1+s}.$$

When the seller and bidders are risk neutral $(s = a = 1)$, then $r_1 = r_2 = r^* = (1+c)/2$. Notice that $r_2 < r^*$ if $s < 1$. For simplicity, assume now that $c = 0$. If $r_1 > 0$ then $r_1$ satisfies

$$(7) \qquad r_1^s = \int_{r_1}^{1} \left[(\frac{m-1}{m}) + (\frac{1}{m})\left(\frac{r_1}{x}\right)^m\right]^{s-1}\left(\frac{r_1}{x}\right)^{m-n} sx^{s-1}dx \,.$$

This expression poses difficulties, so only polar cases will be examined. First, if $a < 1$ then $m-n = (n-1)(1-a)a^{-1} \to \infty$ as $n \to \infty$. Therefore, taking limits in (7) yields $r_1 \to 0$ as $n \to \infty$ if $a < 1$. On the other hand, if $a = 1$ then $m = n$ and $r_1 \to (1/2)^{1/s}$ as $n \to \infty$. Although $(1/2)^{1/s}$ is less than $r_2$, it is certainly greater than zero; it appears that the difference between $r_2$ and $r_1$ is greatest when there are many bidders and they are risk averse, rather than when only the seller is risk averse. If $s = 1$, then either large numbers of bidders or high risk aversion among bidders results in $r_1 = 0$:

$$s = 1 \Longrightarrow r_1 = \begin{cases} 0 & \text{if } 2 \leq m-n = (n-1)(\frac{1-a}{a}) \\ (n-m+2)^{1/(m-n-1)} & \text{otherwise}. \end{cases}$$

6. Ex Ante Comparison

This section addresses the following question: before the auction is held and before the buyers know their values, which type of auction should the seller and the buyers each prefer?

The following lemma is fundamental. It states that whenever the same reservation price is set in the two auctions, the expected sale price will be greater in the first price auction. The two expected sale prices will be equal only if bidders are risk neutral. Vickrey [1961] first proved a special case of this lemma and Butters [1975], Holt [1978], Harris and Raviv [1978], and especially Samuelson [1978] have proved more general cases. An alternative, relatively short proof is sketched here.

Lemma 2: If the same reservation price r is set in both auctions, then $E(w_1|\text{sale}) \geq E(w_2|\text{sale})$. Equality holds if $u''_B = 0$, and strict inequality holds if $u''_B < 0$.

Proof: The following chain utilizes (4*) and holds for $u''_B = 0$.

$$E(w_1|\text{sale})[1-G_n^1(r)] = \int_r^{\bar{v}} b^*(v,r)g_n^1(v)dv$$

$$= \int_r^{\bar{v}} \{\int_r^{v} x(n-1)F(x)^{n-2}f(x)dx + rF(r)^{n-1}\}nf(v)dv$$

$$= \int_r^{\bar{v}} xn(n-1)F(x)^{n-2}f(x) \{\int_x^{\bar{v}} f(v)dv\}dx \quad + \quad rnF(r)^{n-1} \int_r^{\bar{v}} f(v)dv$$

$$= \int_r^{\bar{v}} xg_n^2(x)dx \quad + \quad r[G_n^2(r) - G_n^1(r)]$$

$$= E(w_2|\text{sale})[1-G_n^1(r)].$$

Hence $u_B'' = 0$ implies $E(w_1|\text{sale}) = E(w_2|\text{sale})$. If $u_B'' < 0$, then $b(v,r) > b^*(v,r)$ for all $v > r$, so that the above chain implies

$$\begin{aligned} E(w_1|\text{sale})[1-G_n^1(r)] &= \int_r^{\bar{v}} b(v,r)g_n^1(v)dv \\ &> \int_r^{\bar{v}} b^*(v,r)g_n^1(v)dv \\ &= E(w_2|\text{sale})[1-G_n^1(r)]. \end{aligned}$$

Hence $u_B'' < 0$ implies $E(w_1|\text{sale}) > E(w_2|\text{sale})$. □

An easy corrolary of lemma 2 is that the first price auction is preferred ex ante by the seller if he is risk neutral, and by the bidders if they are risk neutral. A risk neutral seller prefers the first price auction because, by lemma 2, he can achieve expected profits in the first price auction at least as great as in the second price auction by simply setting the reservation price in the first price auction the same as in the second price auction. (Of course, theorem 2 says he can do even better by setting $r_1 < r_2$ if bidders are risk averse.) Risk neutral bidders, on the other hand, prefer the first price auction because $r_1 < r_2$ results in a lower expected sale price in the first price auction. This follows because $r_1 < r_2$ implies $b^*(v,r_1) < b^*(v,r_2)$, so that the expected sale price in the first price auction with $r_1$ is less than the expected sale price in a first price auction with $r_2$, which by lemma 2 is equal to the expected sale price in the second price auction with $r_2$. Without formal proof, proposition 5 summarizes these comments.

Proposition 5: If $u''_B = u''_S = 0$, then both the buyers and the seller are indifferent between the auctions. If $u''_S = 0$ and $u''_B < 0$ (resp. $u''_S < 0$ and $u''_B = 0$), then the seller (resp. buyers) strictly prefer the first price auction.

In many settings both the seller and the buyers will prefer the first price auction, i.e., the first price auction will *ex ante* Pareto dominate the second price auction. A sufficient condition for this is that the risk aversion of bidders be small, and that F satisfy

A3. For any $v > r$, let $b = b^*(v,r)$. Then

$$(n-1)\,\frac{f(b)}{F(b)}\,(v-b) \leq 1.$$

Assumption A3 is similar to the assumption that $b^*_v(v,r) < 1$ (see (3)). The assumption that $b^*_v(v,r) < 1$ is intuitively reasonable and, because $b^*(r,r) = r$ and $b^*(v,r) < v$, must be true on average.[7] If F is the uniform distribution then A3 is satisfied, but in general it is not clear what assumptions on F imply A3. It is clear, however, that A3 is stronger than necessary for the following theorem.

Theorem 2: If A3 holds, then the seller strictly prefers the first price auction whenever $u''_S < 0$ or $u''_B < 0$.

Theorem 2 implies the existence of settings in which the first price auction Pareto dominates *ex ante* the second price auction. Suppose F satisfies A3, and that $u''_S < 0$. Then by theorem 2, the seller strictly prefers the first price auction. By proposition 5, risk neutral bidders also strictly prefer the first price auction. Continuity therefore

implies that given A3 and any $u_S$ with $u_S'' < 0$, the bidders as well as the seller prefer the first price auction if the bidders are not too risk averse.

The proof of Theorem 2 requires a lemma. Assumption A3 is used only in proving this lemma.

Lemma 3: Given $v > r$, let $b = b^*(v,r)$. If $G_n^1(v) = G_n^2(b)$, then $b_v^*(v,r)g_n^2(b) < g_n^1(v)$.

Proof: We have

$$\frac{g_n^1(v)}{g_n^2(b)} = \frac{nF(v)^{n-1}f(v)}{n(n-1)F(b)^{n-2}(1-F(b))f(b)}$$

$$= \frac{G_n^1(v)f(v)}{(n-1)F(b)^{n-2}(1-F(b))F(v)f(b)}$$

$$= \frac{G_n^2(b)f(v)}{(n-1)F(b)^{n-2}(1-F(b))F(v)f(b)}$$

$$= \frac{[nF(b)^{n-1}-(n-1)F(b)^n]f(v)}{(n-1)F(b)^{n-2}(1-F(b))F(v)f(b)}$$

$$= \frac{f(v)F(b)}{F(v)f(b)} \cdot \frac{\frac{n}{n-1} - F(b)}{1 - F(b)}$$

$$> \frac{f(v)F(b)}{F(v)f(b)} \quad .$$

Now, by (3) and A3,

$$b_v^*(v,r) = (n-1)\,\frac{f(v)}{F(v)}\,(v-b)$$

$$= \frac{f(v)F(b)}{F(v)f(b)} \cdot (n-1)\,\frac{f(b)}{F(b)}\,(v-b)$$

$$\leq \frac{f(v)F(b)}{F(v)f(b)}\,.$$

Hence $b_v^*(v,r)g_n^2(b) < g_n^1(v)$. □

<u>Proof of Theorem 2</u>: If $u_S'' = 0$, then by hypothesis $u_B'' < 0$ and the seller prefers the first price auction by proposition 5. Hence we can assume $u_S'' < 0$. We shall prove that the seller prefers the first price auction whenever both auctions have the same reservation price r, which implies that he prefers the first price auction when he can set $r_1 \neq r_2$.

The seller's profit in auction j is

$$z_j = \begin{cases} w_j - c & \text{if a sale occurs} \\ 0 & \text{if no sale occurs.} \end{cases}$$

By a theorem of Hadar and Russell [1969], the seller will prefer the random variable $z_1$ to $z_2$ if $z_1$ strictly stochastically dominates $z_2$ in the second degree.[8] This we show. Let $K_j$ and $k_j$ respectively denote the distribution and density functions of $z_j$. We must show that for any $x \geq 0$,

$$(8) \qquad \int_0^x K_1(z)dz \leq \int_0^x K_2(z)dz,$$

and that strict inequality holds in (8) for some x.

In either auction, the probability of no sale is the probability that $y_n < r$, where $y_n = \max(v_1,\ldots,v_n)$. Also, $z_j$ cannot be between 0 and r-c. Hence,

$$K_1(z) = K_2(z) = G_n^1(r) \quad \text{for } 0 \leq z < r-c.$$

In the case of a sale, $w_2$ is the maximum of r and the second greatest value. Hence

$$K_2(z) = G_n^2(z+c) \quad \text{for } r-c \leq z \leq \bar{v}-c.$$

The sale price in the first price auction is $w_1 = b(y_n,r)$. Therefore, if again we let $\sigma$ be defined by $b(\sigma(p),r) = p$,

$$\begin{aligned} K_1(z) &= \Pr(b(y_n,r)-c \leq z) \\ &= \Pr(y_n \leq \sigma(z+c)) \\ &= G_n^1(\sigma(z+c)) \quad \text{for } r-c \leq z \leq b(\bar{v},r)-c. \end{aligned}$$

Finally, $K_1(z) = 1$ for $z \geq b(\bar{v},r)-c$. (See figure 1.) Let $K_1^*$ be the distribution of $z_1$ when bidders are risk neutral. Since $b^*(v,r) \leq b(v,r)$, $K_1(z) \leq K_1^*(z)$ for any $z \geq r-c$. Hence, instead of showing (8), we need only show that for all $x \geq 0$,

$$(9) \qquad \int_{r-c}^{x} K_1^*(z)\,dz \leq \int_{r-c}^{x} K_2(z)\,dz,$$

and that this inequality is strict for some x.

Let $\bar{z} = \inf\{z \geq r-c \mid K_1^*(z) \geq K_2(z)\}$. Since $K_1^*(r-c) = G_n^1(r) < G_n^2(r) = K_2(r-c)$, $\bar{z} > r-c$. Therefore

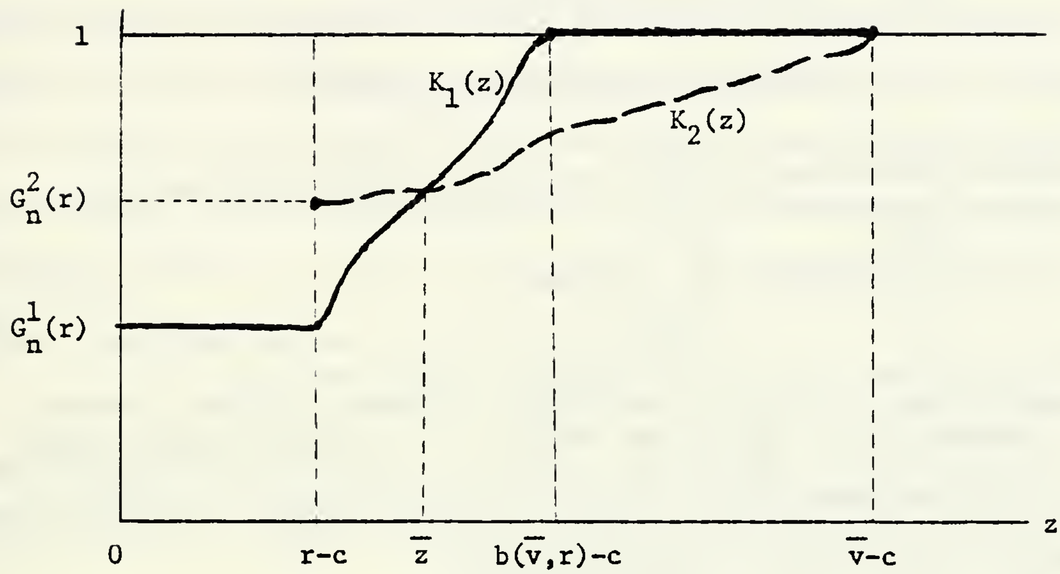


Figure 1

$$\int_{r-c}^{x} K_1^*(z)dz < \int_{r-c}^{x} K_2(z)dz$$

for all $r-c < x < \bar{z}$. Hence it remains only to show (9) for $\bar{z} \leq x \leq \bar{v}$.

Now, lemma 2 and $u_B'' = 0$ imply

$$E(z_1) = E(w_1-c|y_n \geq r)Pr(y_n \geq r)$$

$$= E(w_2-c|y_n \geq r)Pr(y_n \geq r) = E(z_2).$$

Therefore, evaluation of $E(z_j)$ via integration by parts yields

$$\int_{r-c}^{\bar{v}} K_1^*(z)dz = \int_{r-c}^{\bar{v}} K_2(z)dz.$$

Hence (9) holds for $\bar{z} \leq x \leq \bar{v}$ if $K_1^*(z) \geq K_2(z)$ for all $z > \bar{z}$.

Assume $K_1^*(z_0) < K_2(z_0)$ for some $z_0 > \bar{z}$. Therefore there exists $\bar{z} \leq z \leq z_0$ for which $K_1^*(z) = K_2(z)$ and $k_1^*(z) \leq k_2(z)$. As z must be between r-c and $b^*(\bar{v},r)-c$, this implies that

$$g_n^1(\sigma(z+c))\sigma'(z+c) = k_1^*(z) \leq k_2(z) = g_n^2(z+c).$$

Substituting $v = \sigma(z+c)$, $b = b^*(v,r) = z+c$, and $\sigma'(z+c) = 1/b_v^*(v,r)$, we obtain

$$g_n^1(v) \leq b_v^*(v,r)g_n^2(b).$$

This contradicts lemma 3, since $G_n^1(v) = G_n^1(\sigma(z+c) = K_1^*(z) = K_2(z) = G_n^2(z+c) = G_n^2(b)$. Therefore $K_1^*(z) \geq K_2(z)$ for all $z \geq \bar{z}$, so that (9) holds for all $r-c \leq x \leq \bar{v}$. □

## 7. Conclusion

It is widely held that second price auctions are better than first price auctions. One reason, first given by Vickrey [1961], is based on an *ex post* analysis of outcome efficiency.[9] In an example in which the distributions of bidders' values differ, Vickrey finds that in a first price auction the item will sometimes be sold to a bidder who does not value it the most. The first price auction exhibits the same inefficiency in an example of Cox [1978],[10] in which bidders have varying degrees of risk aversion. These authors conclude that second price auctions are preferable, since in a second price auction the item is never sold to a bidder other than the one who values it the most.

However, neither Vickrey nor Cox allow the seller to set different reservation prices in the two auctions. If the seller has this flexibility, theorem 1 states that he sets a lower reservation price in the first price auction, provided he and/or the bidders are risk averse. In some states of the world, therefore, the outcome of the first price auction Pareto dominates the outcome of the second price auction because only in the first price auction does a sale occur. Hence, if agents are risk averse, if the seller sets reservation prices, and if bidders vary in ways other than their realized values, then neither auction is the most efficient in all states of the world.

An *ex post* analysis results in an unambiguous recommendation of one auction on the basis of efficiency only if the outcome of that auction Pareto dominates the outcome of the other auction in every state of the world. This is not the case, even in the simplest setting. One must therefore consider the probabilities attached to various states,

individual attitudes towards risk, and, of course, the equilibrium outcomes of each auction in order to evaluate ex ante welfare.

This approach was taken in section 6. Given an assumption, the seller was shown in theorem 2 to prefer the first price auction, in terms of the expected utility of its outcome, to the second price auction whenever he and/or the bidders are risk averse. This result holds even if the seller must set the same reservation price in both auctions. Since he actually sets a lower reservation price in the first price auction, the buyers also prefer, ex ante, the first price auction in situations where the seller is risk averse and they are approximately risk neutral. Therefore, in a large class of situations the first price auction Pareto dominates the second price auction ex ante.

This result is at first glance contradicted by the results of two recent papers. Harris and Raviv [1978] show, essentially, that within the class of ex ante efficient, two-bidder mechanisms, a second price auction maximizes a risk neutral seller's expected utility.[11] However, their result is derived for risk neutral bidders, which implies, as they point out, that a first price auction yields the same (maximal) expected profit to the seller as does a second price auction. If the bidders are risk averse then the seller's expected profit is greater in the first price auction, as Harris and Raviv also point out. If only the seller is risk averse, then the second price auction is not in the class of ex ante efficient mechanisms, since it can be dominated by the first price auction.

Myerson [1978] also characterizes, within a large class of auctions, an auction that maximizes the seller's expected utility. This auction,

in our framework, is a second price auction. However, Myerson also assumes that the seller and bidders are risk neutral. His result consequently implies that the first price auction, as well as the second price auction, maximizes the seller's expected profit when bidders are risk neutral. If the seller or the bidders are risk averse, the nature of the seller's optimal auction is an open question. But it is not generally a second price auction.

## Footnotes

1. Results are unchanged if several items are to be sold, as long as each buyer purchases at most one item. See Vickrey [1961] and Harris and Raviv [1978].

2. This reservation price r resembles most closely the "cooperatively set minimum prices" used in commodity auctions (Cassady, 1967, p. 230). Evidently these minimum prices are announced by the seller and explicitly set so as to keep sale prices high. Cassady notes that a minimum price is usually set above the price the seller could obtain in a secondary market, which is in accordance with my finding below that the seller sets $r > c$.

3. Cassady [1967, p. 227] reports that often the existence but not the value of a reserve price is announced in oral auctions. (Cassady distinguishes between reserve prices and the minimum prices discussed in footnote 2.)

4. Particular circumstances may facilitate the equilibrium. Bidders' expectations about an unannounced r are more likely to be correct if they know the seller's cost c, since then they can calculate the seller's optimal r. The equilibrium also requires that the seller's threat to not accept bids falling in the interval (c,r) be credible. For an unannounced reservation price, credibility is perhaps enhanced if the seller uses a third-party auctioneer and if he customarily sells in auctions.

5. Samuelson [1978] obtains this result for $u_S'' = 0$.

6. Theorem 1(i) can alternatively be obtained as a corollary to lemma 2 of the next section. Samuelson [1978] obtains 1(i) in this way.

7. It can be shown that $b_v^*(v,r) \leq 1$ if $f'(v) \leq 0$.

8. Hadar and Russell's [1969] theorem is actually for weak preference. A simple modification of their proof yields strict preference when $u_S'' < 0$ and strict second degree stochastic dominance is assumed.

9. Another reason Vickrey [1961] gives for preferring second price auctions is that they, having dominant strategy equilibria, do not give a bidder an incentive to waste resources on gathering and processing information about the behavior of other bidders. This type of information acquisition is studied, in a somewhat different model, in Matthews [1979].

10. Cox attributes the example to John Ledyard.

11. The notion of _ex ante_ used by Harris and Raviv [1978] is different than the one used here. They calculate a bidder's expected utility conditional on his own value, while here it is calculated unconditionally, i.e., from the point of view of a bidder who does not yet know his own value. Independence of the values implies that the two notions of _ex ante_ yield the same result for risk neutral bidders, namely, that they prefer whichever auction has the lower reservation price. Under either definition of _ex ante_, therefore, the first price auction is Pareto superior to the second price auction when the seller is risk averse and the bidders are risk neutral.

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